AN EXTENSION TO CONTINUUM OF THE P.A. SAMUELSON MULTIPLIER-ACCELERATOR ANALYSIS

Preface

P.A. Samuelson's seminal paper "Interactions between the Multiplier Analysis and the Principle of Acceleration" (1939) based on certain assumptions regarding the time lag between, on the one hand, income and consumption and, on the other, investments and consumption, showed, within the framework of a marginal discrete (discontinuous) analysis, the existence of a dynamic process determining national income.

Samuelson demonstrated, with reference to a closed economy, the conditions under which the multiplier and accelerator mechanisms may cause real national income to converge with or diverge from equilibrium income. These movements may be monotonic (regular), or oscillatory.

Contrary to the conclusions reached by the above-mentioned discrete analysis, an extension to continuum of the same marginal analysis performed by maintaining the original reference to a closed economy, appears to demonstrate an evolution of the national income which always *diverges* with regularity from the equilibrium income value. Moreover, the closer to zero the marginal propensity to consume is, the more important is the resultant divergence that will occur in respect of the equilibrium income value.

1. A brief recall of the Samuelson models and of its relevant conclusions[1]

The assumptions considered in the Samuelson model are the following.

In each definite period t the national income Yt is formed by the consumption expenditure $Ct,$ the induced private investment $It`$ and the autonomous investment (governmental expenditure) assumed to remain constant $It`` = G$.

[1] See: Samuelson (1939, 1947). On the multiplier principle see also Chipman (1951).

The consumption depends upon the national income Yt according to the following function:

$$Ct = \alpha Y\ t\text{-}1 \tag{1}$$

where:

Ct: consumption

Y: national income

α: marginal propensity to consume, with values from 0 to 1.

Besides, the induced investments depend on the variations in the demand for consumption goods according to the acceleration principle:

$$It` = \beta\ (Ct - C\ t\text{-}1) \tag{2}$$

where β is the acceleration coefficient ($\beta > 0$) (the *relation of acceleration* according to Alvin Hansen terminology).

The following condition of equilibrium

$$Yt = Ct + It \tag{3}$$

ends the model, where

$$It = It` + It`` \tag{4}$$

which gives:

$$Yt = Ct + It` + G \tag{5}$$

With a series of simple substitutions we then obtain the following second order equation:

$$Yt = \alpha Y\ t-1 + \beta\ [\alpha Y\ t - 1 - \alpha Y\ t - 2] + G \tag{6}$$

which may be also described as follows :

$$Yt - \alpha\ (1+\beta)\ Y\ t\text{-}1 + \alpha\ \beta Y\ t\text{-}2 = G \tag{7}$$

We then continue to a finite difference equation of the second order whose general solution Yt is the sum of a particular solution which indicates the value of equilibrium of the national income (Ye) and of a general solution Yh of the homogenous equation (i.e. an equation without the second member) which may be interpreted as being the difference between the real national income and the equilibrium income.

More precisely, we can easily obtain a particular solution for the equation set-out in (7) above by setting Yt = constant and getting the equilibrium value Ye of the national income:

$$Ye = \frac{G}{1-\alpha} \tag{8}$$

where the income is set in relation to the autonomous investment G through the Keynesian type multiplier $\frac{1}{1-\alpha}$.

The divergence from the equilibrium value will then be derived by the general solution of the homogenous equation corresponding to (7) above, i.e.:

$$Yt - \alpha\ (1+\beta)\ Y\ t\text{-}1 + \alpha\ \beta Y\ t\text{-}2 = 0 \tag{9}$$

This general solution of the homogenous equation is of the following type:

$$Yh = C1b1^{t} + C2\ b2^{t} \tag{10}$$

if the roots *b1* and *b2* of the characteristic equation

$$b^{2} - \alpha\ (1 + \beta)b + \alpha\ \beta = 0 \tag{11}$$

happen to be real and distinct and of the type

$$Yh = C3b^{t} + C4tb^{t} = b^{t}\ (C3 + C4t) \tag{12}$$

if the characteristic equation possesses a double root (*b*).

In the case of *complex* roots, which may be indicated as *b1, b2* $= u \pm v^{1}$, the solution of the homogeneous equation may be written as follows:

$$Yh = R^{t}\,(C5\ \cos\ \mu t + C6\ \sin\ \mu t\ ) \tag{13}$$

where R represents the module of the complex roots, i.e.

$\sqrt{u2 + v2}$

and μ is an angle measured by radiants, so that

$$\cos\mu = \frac{u}{R} \tag{14}$$

and

$$\sin\ \mu = \frac{v}{R} \tag{15}$$

i.e.

$$\mu = arctg\ v/u \tag{16}$$

If we now pass to a quantitative analysis of (11) and if we introduce the stability conditions[2], we obtain:

$$1 - \alpha\ (1 + \beta) + \alpha\ \beta = 1 - \alpha > O \tag{17}$$

$$1 - \alpha\ \beta > O \tag{18}$$

$$1 + \alpha\ (1 + \beta) + \alpha\ \beta > 0 \tag{19}$$

The first inequality is entirely satisfied, as we presume that the marginal propensity to consume is inferior to one; similarly, the third inequality is also equally satisfied as in the first member we have a sum of quantities that all happen to be positive.

[2] These stability conditions form the necessary and sufficient conditions for a convergent oscillation. The (17), (18), (19) form the necessary and sufficient condition in order that the roots of the characteristic equation (11) – real or complex – be in absolute value inferior to unity. On the matter see Samuelson (1941) and Gandolfo (1971).

The crucial inequality that results is the second one. We may therefore deduce that the stability condition is the following:

$$\alpha \beta < 1 \tag{20}$$

or

$$\alpha < 1 / \beta \tag{21}$$

In order to establish the type of movement (monotonic, oscillatory, etc.), we must observe at first that the succession of the signs of the equation (11) is +, –, +. This means that according to the Cartesius signs rule there will be no negative roots. We may then exclude the existence of movements bringing about 'unproper' oscillations.

The following step will be that of calculating the 'discriminant' of the (11).

It will be:

$$\Delta = \alpha^2 (1+\beta)^2 - 4 \alpha \beta = \alpha [\alpha (1+\beta)^2 - 4 \beta] \tag{22}$$

Its sign will indicate if the roots are real and distinct, double or complex.

As α *is positive,* we may distinguish the following cases:

Δ will be $> O$, $= O$, or $< O$ if respectively $\alpha (1+ \beta)^2$ will be $>$, $=$, or $< 4 \beta$ (23)

which gives:

$\Delta > O$, $= O$, or $< O$ if respectively α will be $>$, $=$, or $< 4 \beta/(1+ \beta)^2$ (24)

By considering together the (21) and the (24) we have all the possible cases which may be properly illustrated by referring to the well known Samuelson graphic representation.

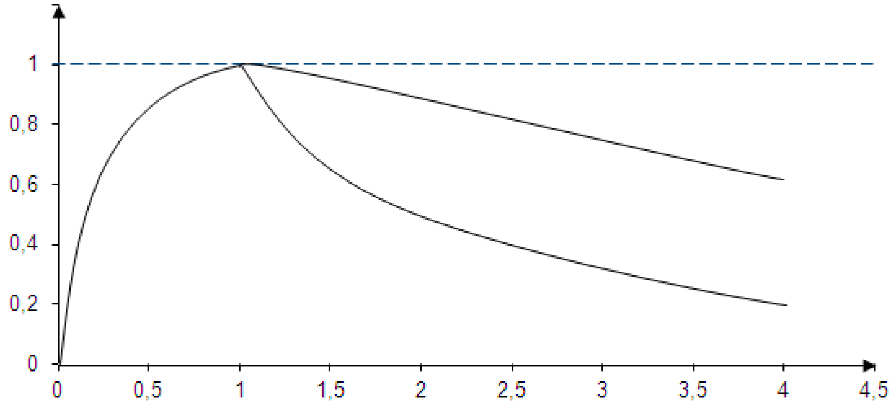
We hint at the Chart 2 of his already mentioned contribution: "Diagram showing Boundaries of Regions yielding Different Quantitative Behavior of National Income".

When we refer to Samuelson diagram, we find the description of the function

$\alpha = 4 \beta / (1+ \beta)^2$ (indicated in our reproduction set-out below, as the OPS curve) and of the function $\alpha = 1/ \beta$ (described as the curve PQ). As $\alpha < 1$, we are interested only in the part of the positive quadrant that remains beneath the *broken line*. It explains why we may then neglect the superior part of the hyperbola $\alpha = 1/\beta$.

We then detect the same four Regions, as already described by Samuelson, by bearing in mind the fact that the real roots are positive.

DIAGRAM 1



Region A

Every point of this region lies under the function $\alpha = 1/\beta$ and above the function

$\alpha = 4\,\beta\,/\,(1+\beta)^2$, as a consequence at the graphical level of the two abovementioned inequalities

$\alpha < 1/\,\beta$, $\alpha > 4\,\beta/\,(1+\beta)^2$.

Therefore, following (21) and (24) above, the stability condition is satisfied and the roots are real.

The system will reveal a monotonic (regular) movement converging (*monotonic converging movement*) towards the equilibrium value $Ye = \dfrac{G}{1-\alpha}$.

Region B

Every point of this region satisfies the inequalities

$\alpha < 1/\,\beta$, $\alpha < 4\,\beta/\,(1+\beta)^2$.

The stability condition is satisfied and the roots are complex. The result will be a convergence with oscillations (*damped oscillation*) around the equilibrium value.

Region C

In this region we have $\alpha > 1/\,\beta$, $\alpha < 4\,\beta/\,(1+\beta)^2$.

The stability condition in this case is *not* fully satisfied and the roots are complex. The result will be a divergence with oscillations (*explosive oscillation*) around the equilibrium value.

Region D

In this region we have $\alpha > 1/\beta$, $\alpha > 4\beta/(1+\beta)^2$.

The stability condition is therefore *not* satisfied and the roots are real. The movement is a regular divergence from the equilibrium value (*monotonically explosive*).

For completion of the above analysis, we may also examine the points lying on the curves which form the separation lines of the Regions. As both α and β remain, according to the model under examination, positive, and given that $\alpha < 1$, we may therefore neglect the points lying on the axes α and β, on the axes origin and on the point P.

We then have the following particular cases.

1.1 Points on the Separation Line between Regions A and B

Here we have the formula $\alpha < 4\beta/(1+\beta)^2$: therefore there will exist a double real root. As the stability condition $\alpha < 1/\beta$ is satisfied, this root will be inferior to 1. As a consequence the function

$b^t\ (C3+C4t)$

will be dominated for t tending to infinity by the component b^t and the movement *will converge regularly towards the equilibrium value*.

1.2 Points on the Separation Line between Regions B and C

Here we have:

$\alpha < 4\beta/(1+\beta)^2$ and $\alpha\beta = 1$. In this case the roots remain complex with a unitary module, which implies *an oscillation of a constant width*, i.e. a movement separating the stability from the instability.

1.3 Points on the Separation Line between Regions C and D

Here we have $\alpha = 4\beta/(1+\beta)^2$ and $\alpha > 1/\beta$. Therefore, we have a double root superior to 1. The result will be a movement that is eventually dominated by the *regularly divergent term b^t*.

2. An extension to continuum of the P.A. Samuelson model

Our basic idea has been to verify whether the Samuelson model would have arrived at the same conclusions if the marginal analysis had not been of a discontinuous (discrete) type, but instead of a continuum type.

As a preliminary to the analysis of a continuum type that we attempt to perform, we need to consider the national income *y* at the time *t*, indicated as *y(t),* defined in the following manner:

$$\int_{t1}^{t} y\ (t)dt = Y(t2) - Y(t1) \tag{25}$$

where

Y(t2) – Y(t1) represents the income generated during the generic interval t1– t2[3].

By adapting hypothesis 1. and 2. of the Samuelson model to a continuum analysis, these then respectively become[4]:

$$C\ (t) = \alpha\ [\int_{t1}^{t} y(\mathrm{T})dT + Y(t1)] \tag{26}$$

and

$$I(t) = \beta\ C`(t) \tag{27}$$

The (5) is then translated into:

$$Y(t) = \alpha \int_{t1}^{t} y(T)dT + \alpha\ Y\ (\ t1) + \beta C'\ (t) + G. \tag{28}$$

From (26) we obtain for the Torricelli theorem the following derivative:

$$C`(t) = \alpha\ y(t) \tag{29}$$

By substituting this in (28) we then have, moreover:

$$Y(t) = \alpha + \int_{t1}^{t} y(T)dT + \alpha\ Y\ (t1) + \alpha\ \beta y\ (t) + G. \tag{30}$$

By deriving both the members of (30) and remembering what was indicated in footnote 3, i.e. that $y(t) = Y`(t)$ – we then have*:*

$$y(t) = \alpha\ y(t) + \alpha\ \beta y`\ (t) \tag{31}$$

therefore

$$\alpha\ \beta y`\ (t) - (1 - \alpha)y(t) = 0 \tag{32}$$

and, moreover, as being $y(t) = Y`(t)$, then $y`(t) = Y``\ (t)$, we finally have at the end:

$$\alpha\ \beta Y``\ (t) - (1 - \alpha)Y`(t) = 0. \tag{33}$$

[3] This definition results from the need to create a correlation between the 'instantaneous' income, i.e. that produced at the instant t, y(t), and the income produced prior to the time t. The (25) referred to the interval from t1 to t then gives: $\int_{t1}^{t} y(T)dT$ = Y(t) – Y(t1) from which we have by deriving both members y(t) = Y`(t).

[4] We presume that Y(t), y(t), C(t) and I(t) are continuous functions that are derivable from time t and we will indicate the first and second derivatives with the customary notations used in the mathematical analysis.

The (33) represents a differential equation of second order whose integral is of the type:

$$Y(t) = C1e^{b1t} + C2e^{b2t} \tag{34}$$

where b1 and b2 are the roots of the characteristic equation

$$\alpha\beta\, b^2 - (1 - \alpha)\mathrm{b} = 0 \tag{35}$$

and precisely

$$b1 = 0;\ b2 = \frac{1-\alpha}{\alpha\beta} \tag{36}$$

It therefore follows that the roots always remain real; moreover, being $\alpha < 1$, they are also distinct. We may then forecast that the trend of the actual national income will be univocally divergent from the equilibrium income, irrespective of what the values assumed by α and β may be.

Consequently, (34) as general solution for (33) becomes:

$$Y(t) = C1 + C2\ e^{[1-\alpha/\alpha\beta]t} \tag{37}$$

which depends, as we happen to talk of a second order differential equation, on two constants C1 and C2 that we may determine by imposing the conditions to the drawn contour Y(0) and Y`(0), i.e. by attributing the values

$$\mathrm{Y}(0) = \mathrm{C1} + \mathrm{C2} \tag{38}$$

and

$$\mathrm{Y}`(0) = \mathrm{C2}\ \frac{1-\alpha}{\alpha\beta}. \tag{39}$$

In fact, one of the constants may be determined by the equation (30), by introducing a constant in the time Y. According to this assumption we then have:

$$Y(t) = Y(t1) = Y(0),\ \text{then}\ Y`(t) = y(t) = 0 \tag{40}$$

Therefore, (30) gives rise (by setting t = 0) to

$$Y(0) = \alpha\ Y(0) + G \tag{41}$$

and, finally, at the end:

$$\mathrm{Y}(0) = \frac{G}{1-\alpha} \tag{42}$$

Here we again find the equilibrium value of the national income that is considered in the Samuelson analysis – see part (8). By substituting part (42) in (38) we then have:

$$\frac{G}{1-\alpha} = \mathrm{C1} + \mathrm{C2}. \tag{43}$$

Thus

$$C1 = \frac{G}{1-\alpha} - C2, \tag{44}$$

which, when substituted in the general solution of the differential equation (37) gives:

$$Y(t)= \frac{G}{1-\alpha} - C2 + C2\ e^{[1-\alpha/\alpha\beta]t} \tag{45}$$

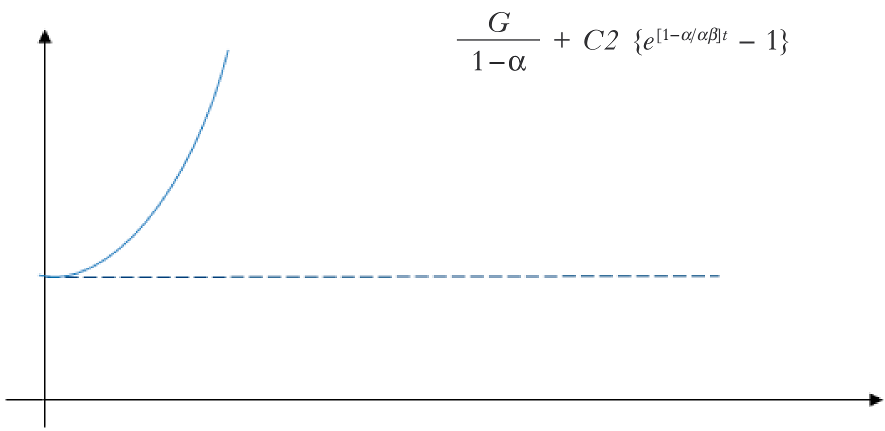
and, finally at the end:

$$Y(t)= \frac{G}{1-\alpha} + C2\{e^{[1-\alpha/\alpha\beta]t} - 1\}$$

From this function we may then deduce that if t tends to 0, then $Y(t)$ tends to $\frac{G}{1-\alpha}$, whereas if t tends to infinity, then the Y(t) will regularly diverge from the equilibrium income.

This trend may be illustrated by the following graphic representation.

DIAGRAM 2



In this diagram, the initial inclination of the function Y depends on the un-named constant – not yet determined – C2 according to (39), whereas the opening of the curve depends on the value $\frac{1-\alpha}{\alpha\beta}$ which appears in the exponential, whose value increases as α tends to zero.

Finally, if the marginal propensity to consume remains near to zero, then the actual national income will diverge rapidly (i.e. in a regular manner), whereas, when α tends to one, the actual income will start to diverge more and more slowly from the equilibrium value.

3. Conclusion

As illustrated above, Samuelson's marginal discrete (discontinuous) analysis shows, according to the values respectively assumed by α and β, a trend of a convergent or divergent nature (regular or with oscillations) of the real national income with regard to the equilibrium value $G/(1-\alpha)$.

On the contrary, an extension to continuum of the same analysis would reveal (with reference to the same value of the equilibrium income), a trend whereby real national income always remains divergent, independent of the values respectively assumed by α and β. This divergent trend will also occur even faster, the closer to zero the value of the marginal propensity to consume happens to be.

This analysis seems then to prove that in a closed economy, the multiplier and accelerator mechanisms affect the evolution of national income in a way far different from what was established within the framework of a marginal discrete analysis. The permanently divergent trend of national income with regards to equilibrium income leads to the conclusion that in a closed economy the equilibrium conditions of national income continue to be much more difficult to achieve than was previously assumed by Samuelson's discrete analysis.

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ABSTRACT

The aim of this article is to demonstrate that the conclusions drawn by Samuelson's marginal discrete analysis in his 1939 seminal paper "Interactions between the Multiplier Analysis and the Principle of Acceleration" concerning the equilibrium conditions of real national income in a closed economy are not valid anymore if the same equilibrium conditions are examined with the framework of a marginal continuous analysis.

Samuelson's discrete analysis shows a trend of converging or divergent nature (monotonic or oscillatory) of real national income with regard to the equilibrium income value, according to the values respectively assumed by the multiplier and the accelerator.

Contrary to the conclusions reached by the above-mentioned discrete analysis, an extension to continuum of the same analysis, always referring to a close economy, would reveal a permanently divergent trend of real national income with regard to the equilibrium value.

This divergent trend will also occur even faster, the closer to zero the marginal propensity to consume is.

The marginal continuous analysis then seems to prove that in a close economy the equilibrium conditions of national income continue to be much more difficult to achieve than was previously assumed by Samuelson's discrete analysis.

RIASSUNTO

Una estensione al continuo del modello del moltiplicatore-acceleratore di Samuelson

Lo scopo di questo articolo è quello di dimostrare che le conclusioni raggiunte da P.A. Samuelson, sulla base di una analisi di tipo discreto, nel suo contributo del 1939 "Interactions between the Multiplier Analysis and the Principle of Acceleration" a proposito delle condizioni di equilibrio del reddito nazionale in una economia chiusa perdono la loro validità se le stesse condizioni vengono esaminate secondo una analisi marginale di tipo continuo.

La analisi di tipo discreto condotta da Samuelson, nell'ambito appunto di una economia chiusa, mostra una evoluzione del reddito effettivo rispetto al reddito di equilibrio convergente ovvero divergente, di natura monotona (regolare) o oscillatoria, a seconda dei valori rispettivamente assunti dal moltiplicatore e dall'acceleratore.

Diversamente dalle conclusioni raggiunte da Samuelson in campo discreto, una estensione al continuo della stessa analisi, sempre con riferimento ad una economia chiusa, indicherebbe invece una evoluzione del reddito sempre comunque divergente con regolarità rispetto allo stesso valore di equilibrio dedotto nella analisi discreta.

Tale risulterebbe poi tanto più marcata quanto più prossimo allo zero risulti il valore della propensione marginale al consumo.

L'estensione al continuo della analisi proverebbe quindi che in una economia chiusa le condizioni di equilibrio del reddito nazionale restano molto più difficili da realizzare di quanto prima supposto secondo la analisi di tipo discreto proposta da Samuelson.

Keywords: Closed Economy, Real National Income, Equilibrium Conditions, Convergent and Divergent Trends

JEL Classification: D5, D50