Authors:
Abdulnasser Hatemi-J
Department of Accounting and Finance, College of Business and Economics, UAE University, Al Ain, The United Arab Emirates
Youssef El-Khatib
Department of Mathematical Sciences, UAE University, Al Ain, The United Arab Emirates

THE DIVIDEND DISCOUNT MODEL WITH MULTIPLE GROWTH RATES OF ANY ORDER FOR STOCK EVALUATION

Abstract

The dividend discount model (DDM) is regularly used for stock evaluations. However, in the existing literature the solution of this model exists only for a couple of different growth rates. The current paper provides a general solution for the DDM to compute the intrinsic value of a common stock that allows for multiple stage growth rates of any predetermined number of periods. A mathematical proof is provided for the suggested general solution. A numerical application is also presented. The solution introduced in this paper is expected to improve on the precision of stock valuation, which might be of fundamental importance for the individual investor as well as for financial institutions.

Keywords: Stock Evaluation; The Dividend Discount Model; Multiple Growth Rates; Closed Form Solution
JEL Classification: G12; C00

Riassunto

Il modello di sconto dei dividendi con tassi di crescita multipli
di qualsiasi ordine nelle valutazioni azionarie

Il modello di sconto dei dividendi è regolarmente utilizzato nelle valutazioni azionarie. Tuttavia, nella letteratura esistente questo modello è risolto solo con riferimento ad un paio di tassi di crescita diversi. Questo studio fornisce una soluzione generale per tale modello affinché si possa calcolare il valore intrinseco di un pacchetto di azioni ordinarie che tenga conto di tassi di crescita multipli per qualsiasi predeterminato numero di periodi. A supporto di questa soluzione generale viene fornita una dimostrazione matematica. Insieme ad essa si presenta un'applicazione numerica. Si ritiene che la soluzione proposta in questo articolo migliori la

precisione della valutazione delle azioni, fatto di fondamentale importanza sia per l'investitore privato che per quello istituzionale.

1. INTRODUCTION

Determining a measure that can represent the intrinsic value of a stock is an important issue for investors and financial institutions seeking profitable investment prospects. This issue might be also of interest for policy makers in order to design appropriate financial guidance. The dividend discount model (DDM), which was originally been developed by Gordon and Shapiro (1956) and Gordon (1959, 1962), can be used for this purpose[1]. Currently, there are extensions of the model in the literature that allow for the valuation of a common stock with two different growth rates across time at the maximum, to our best knowledge[2]. In this paper we suggest a general solution for the valuation of common stocks for the DDM that allows for multiple growth rates of any predetermined number. The suggested solution is proved mathematically, and an application is provided for a case with three different growth rates, and the financial implication of the findings is outlined.

The remaining part of this paper is organized as follows. The next section presents the model. Section 3 applies the suggested solution to valuate a common stock with three different growth rates. The last section offers the concluding remarks. A mathematical proof for the general solution is provided in the appendix, which is presented at the end.

2. THE MODEL WITH MULTIPLE GROWTH RATES FOR STOCK EVALUATION

We consider N different growth annuity periods T_1 with growth g_1, T_2 with growth g_2, $\cdots$, and T_N with growth g_N, followed by a perpetuity with growth rate g_{N+1}. For $1 \leq l \leq N$, we denote by $ST_l := \sum_{i=0}^{i=l} T_i$ the end of period T_l. The time is to be going from $t = T_0 := 0$ to $ST_N := \sum_{i=0}^{i=N} T_i$ with N annuities and from $ST_N + 1$ to ∞ with a perpetuity as explained in the following:

During $T_0 \to T_1 = ST_1$ the growth rate is g_1,

[1] See also Gordon and Gould (1978) as well as Fuller and Hsia (1984). For additional publications on stocks see, among others, Hatemi-J (2019), Hatemi-J and Al Mohana (2019), Dimingo *et al.*, (2021), Darmawan *et. al.*, (2021) and Hatemi-J and Taha (2021).

[2] It should be mentioned that there are also alternative models for stock valuation such as the capital asset pricing model developed by Sharpe (1964) and the arbitrage pricing theory suggested by Ross (1976).

The dividend at year i before ST_1 is $D_i = D_0(1+g_1)^i$

The dividend at the end of the period is $D_{ST_1} = D_0(1+g_1)^{T_1}$

During $T_1 + 1 = ST_1 + 1 \rightarrow T_1 + T_2 = ST_2$ the growth rate is g_2,

The dividend at year i between ST_1 and ST_2 is $D_i = D_{ST_1}(1+g_2)^i$

The dividend at the end of the period is $D_{ST_2} = D_{ST_1}(1+g_2)^{T_2}$

Continuing recursively until the following:

During $ST_{k-1} + 1 \rightarrow ST_k$ the growth rate is g_k,

The dividend at year i between ST_{k-1} and ST_k is $D_i = D_{ST_{k-1}}(1+g_k)^i$

The dividend at the end of the period is $D_{ST_k} = D_{ST_{k-1}}(1+g_k)^{T_k}$

And similarly by continuing we have

During $ST_{N-1} + 1 \rightarrow ST_N$ the growth rate is g_N,

The dividend at year i between ST_{N-1} and ST_N is $D_i = D_{ST_{N-1}}(1+g_N)^i$

The dividend at the end of the period is $D_{ST_N} = D_{ST_{N-1}}(1+g_N)^{T_N}$.

Moreover, for any $1 \leq k \leq N$ and $0 \leq l \leq T_{k+1}$ we can express the following:

$$D_{ST_k+l} = D_{ST_k}(1+g_{k+1})^l = \left[D_0 \prod_{i=0}^{k}(1+g_i)^{T_i}\right](1+g_{k+1})^l. \tag{1}$$

The dividend after the $ST_N + m$ years is $D_{ST_N+m} = D_{ST_N}(1+g_{N+1})^m$.

Now we can present our main result via the following proposition.

Proposition 1 The value of a share, P_0, is given by

$$P_0 = \sum_{k=1}^{N+1} \frac{D_{ST_{k-1}+1}}{(1+r)^{ST_{k-1}}(r-g_k)} A_k, \tag{2}$$

where

$$A_k := \left[1 - \left(\frac{1+g_k}{1+r}\right)^{T_k}\right], \qquad \text{for any } 1 \le k \le N, \tag{3}$$

$$A_{N+1} := 1. \tag{4}$$

Note that the solution expressed in equation (2) is applicable if and ony if $r \neq g_i \ \forall i$. For the proof of this proposition see the appendix.

It should be mentioned here we assume that the growth rates are known *a priori*. However, the practitioners need to specify these values based on the information they have using their subjective probabilities. For example, it could be based on different market conditions such as a financial crisis, a pandemic, a pandemic combined with a war situation, regular market conditions, good economic conditions or extremely bad or good market conditions. The expected dividend growth rates are going to be different in these different market conditions. The better is the market condition, the higher is the expected dividend payout rate going to be under the *ceteris paribus* hypothesis.

In the previous proposition, we provide the value of a share as a function of different growth rates g_i and dividends D_{ST_i} at end of period ST_i, for the interest rate r. We can derive a more explicit formula as a function of D_0, with different growth rates g_i and the interest rate r as is shown in the following corollary.

Corollary 1 *The value of a share* P_0 *is given by*

$$P_0 = D_0 \sum_{k=1}^{N+1} \frac{\prod_{i=0}^{k-1} (1+g_i)^{T_i} (1+g_k)}{(1+r)^{ST_{k-1}} (r-g_k)} A_k, \tag{5}$$

where A_k, for $1 \le k \le N+1$, are given by equations (3) and (4).

Proof. The proof is straightforward by embedding the value of $D_{ST_{k-1}+1}$ from (1) into the formula (2).

Example 1 In this example, we consider the case of $N=1$, i.e. when there is one growth rate g_1 at time T and after that a perpetuity with growth rate g_2. The time is then going from $T_0 = 0$ to $T_1 = T$ with a growth rate g_1 and after T with growth rate g_2. The formula (9) for this particular case can be written as

$$P_0 = \frac{D_0(1+g_1)}{(r-g_1)} A_1 + \frac{D_0(1+g_1)^{T_1}(1+g_2)}{(1+r)^{ST_1}(r-g_2)} A_2 = \frac{D_1}{(r-g_1)}\left[1-\left(\frac{1+g_1}{1+r}\right)^T\right] + \frac{D_{T+1}}{(1+r)^T(r-g_2)}.$$

Example 2 In this example, we consider the case of $N=2$, i.e. we have annuities with growth rates g_1 for time T_1, g_2 for time T_2 and after that a perpetuity with growth rate g_3. The time is then going from $T_0 = 0$ to T_1 with a growth rate g_1, from $T_1 + 1$ to $T_1 + T_2$ with a growth rate g_2 and after $T_1 + T_2$ with the growth rate g_3. Using formula (2) and equations (3-4) we have

$$P_0 = \sum_{k=1}^{3} \frac{D_{ST_{k-1}+1}}{(1+r)^{ST_{k-1}}(r-g_k)} A_k \quad = \frac{D_{ST_0+1}}{(1+r)^{ST_0}(r-g_1)} A_1 + \frac{D_{ST_1+1}}{(1+r)^{ST_1}(r-g_2)} A_2 + \frac{D_{ST_2+1}}{(1+r)^{ST_2}(r-g_3)} A_3,$$

where

$$ST_0 = 0, \quad ST_1 = T_1, \quad \text{and} \quad ST_2 = T_1 + T_2,$$

and

$$A_1 = \left[1-\left(\frac{1+g_1}{1+r}\right)^{T_1}\right], \; A_2 = \left[1-\left(\frac{1+g_2}{1+r}\right)^{T_2}\right], \; A_3 = 1.$$

Thus, the following can be expressed:

$$P_0 = \frac{D_1}{(r-g_1)}\left[1-\left(\frac{1+g_1}{1+r}\right)^{T_1}\right] + \frac{\frac{D_{T_1+1}}{(r-g_2)}}{(1+r)^{T_1}}\left[1-\left(\frac{1+g_2}{1+r}\right)^{T_2}\right] + \frac{\frac{D_{T_1+T_2+1}}{(r-g_3)}}{(1+r)^{T_1+T_2}}.$$

Example 3 In this example, we consider the case of $N = 3$ i.e. we have annuities with growth rates g_1 for time T_1, g_2 for time T_2, g_3 for time T_3 and after that a perpetuity with the growth rate g_4. Again, using formula (2) and equations (3-4) we have

$$\begin{aligned} P_0 &= \sum_{k=1}^{4} \frac{D_{ST_{k-1}+1}}{(1+r)^{ST_{k-1}}(r-g_k)} A_k \\ &= \frac{D_{ST_0+1}}{(1+r)^{ST_0}(r-g_1)} A_1 + \frac{D_{ST_1+1}}{(1+r)^{ST_1}(r-g_2)} A_2 \\ &+ \frac{D_{ST_2+1}}{(1+r)^{ST_2}(r-g_3)} A_3 + \frac{D_{ST_3+1}}{(1+r)^{ST_3}(r-g_4)} A_4, \end{aligned}$$

where

$$ST_0 = 0, \quad ST_1 = T_1, \quad ST_2 = T_1 + T_2, \quad ST_3 = T_1 + T_2 + T_3$$

and

$$A_1 = \left[1 - \left(\frac{1+g_1}{1+r}\right)^{T_1}\right], \; A_2 = \left[1 - \left(\frac{1+g_2}{1+r}\right)^{T_2}\right], \; A_3 = \left[1 - \left(\frac{1+g_3}{1+r}\right)^{T_3}\right], \; A_4 = 1.$$

Thus, we have

$$P_0 = \frac{D_1}{(r-g_1)}\left[1 - \left(\frac{1+g_1}{1+r}\right)^{T_1}\right] + \frac{\frac{D_{T_1+1}}{(r-g_2)}}{(1+r)^{T_1}}\left[1 - \left(\frac{1+g_2}{1+r}\right)^{T_2}\right] + \frac{\frac{D_{T_1+T_2+1}}{(r-g_3)}}{(1+r)^{T_1+T_2}}\left[1 - \left(\frac{1+g_3}{1+r}\right)^{T_3}\right] + \frac{\frac{D_{T_1+T_2+T_3+1}}{(r-g_4)}}{(1+r)^{T_1+T_2+T_3}}.$$

For the cases when $N > 3$, the closed form solution can be derived similarly.

3. An application

Suppose that a bank has just paid a dividend per share of \$2. Its dividend is expected to grow at 5% during the 3 forthcoming years, it is expected to grow by 7% in years 4 to 7 and after that it is expected to grow at 6% per year in perpetuity. The required rate of return is assumed to be 9%. This information means that the DDM with three growth rates can be used in order to find the intrinsic value of a share for this corporation.

Based on formula (2) we can express the following:

$$P_0 = \frac{D_1}{r - g_1}\left[1 - \frac{(1+g_1)^{T_1}}{(1+r)^{T_1}}\right] + \frac{\frac{D_{T_1+1}}{r - g_2}\left[1 - \frac{(1+g_2)^{T_2}}{(1+r)^{T_2}}\right]}{(1+r)^{T_1}} + \frac{\left(\frac{D_{T_1+T_2+1}}{r - g_3}\right)}{(1+r)^{T_1+T_2}}$$

TABLE 1 - *The Application*

Variable	Value	Variable	Value
D_0	2	r	0.09
D_1	2.1	T_1	3
g_1	0.05	T_2	4
g_2	0.07	D_{T_1+1}	2.47732
g_3	0.06	$D_{T_1+T_2+1}$	3.21691
		P_0	71.05809

Notes: D_i is the dividend in year i and g_j is the growth rate at period j. T_b means break period at time b. P_0 represents the common stock value based on the suggested model.

As it is evident from Table 1, the present value of the underlying common stock is $71.05809 based on our solution. This solution can be used as a benchmark price in order to compare it with the real market price for evaluating whether the price is correctly determined or not. If the stock is not fairly priced, regardless of if it is overpriced or underpriced, this information can be used for finding a strategy that results in arbitrage by choosing an appropriate long or short position in the underlying stock. The existence of the arbitrage opportunity would in turn imply that the market for the underlying stock is not informationally efficient.

4. CONCLUSIONS

Stock valuation is an important issue in financial markets for financial actors. This seems to be the case for both domestic as well as the foreign ones due to increasing globalization. One model that is regularly used for this purpose is the dividend discount model (DDM). Currently, this model allows for only two different growth rates to the best of our knowledge. In this paper we suggest a general solution for the DDM that can handle multiple growth rates of any potential number. The suggested solution is proved mathematically. A numerical example is provided in order to apply the suggested model for a case in which there are three different growth rates of the dividend payouts. This estimation has important financial implications because the investor can determine whether the underlying stock is correctly priced or not in the real market. The result has also implications in terms of arbitrage and market inefficiency.

The closed form solution introduced in this paper is expected to improve on the precision of stock valuation for many different economic conditions, which might be of fundamental importance for investors, financial institutions as well as policy makers.

REFERENCES

Darmawan, I., H. Siregar, D. Hakim and A. Manurung (2021), "Crude Oil Price Movement and Stock Market Trading Activity: Evidence from Indonesia", *Economia Internazionale/International Economics*, 74(1), 25-46.

Dimingo, R., J.W. Muteba Mwamba and L. Bonga-Bonga (2021), "Prediction of Stock Market Direction: Application of Machine Learning Models", *Economia Internazionale/International Economics*, 74(4), 499-536.

Fuller, R.J. and C-C. Hsia (1984), "A Simplified Common Stock Valuation Model", *Financial Analysts Journal*, 40(5), 49-56.

Gordon, M.J. (1959), "Dividends, Earnings and Stock Prices", *Review of Economics and Statistics*, 41(2), 99-105.

Gordon, M.J. (1962), The Investment, Financing, and Valuation of the Corporation, R.D. Irwin: Homewood, IL.

Gordon, M.J. and Gould, L.I. (1978), "The Cost of Equity Capital: A Reconsideration", *Journal of Finance*, 33(3), 849-861.

Gordon, M.J. and E. Shapiro (1956), "Capital Equipment Analysis: The Required Rate of Profit", *Management Science*, 3(1), 102-110.

Hatemi-J, A. (2019), "The Causal Impact of Stock Market Development on Economic Development in the UAE: An Asymmetric Approach", *Economia Internazionale/International Economics*, 72(2), 171-184.

Hatemi-J, A. and S. Al Mohana (2019), "Testing for Financial Market Integration of the UAE Market with the Global Market", *Economia Internazionale/International Economics*, 72(4), 475-492.

Hatemi-J, A. and V. Taha (2021), "Portfolio Diversification Benefits between Financial Markets of the US and China: Empirical Evidence from Two Alternative Methods", *Economia Internazionale/International Economics*, 74(4), 537-546.

Ross, S. (1976), "The Arbitrage Theory of Capital Asset Pricing", *Journal of Economic Theory*, 13(3), 341-360.

Sharpe, W.F. (1964), "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk", *Journal of Finance*, 19(3), 425-442.

APPENDIX

Proof of Proposition 1. First, notice that the value of a share is given –in general terms– by the series

$$P_0 = \sum_{i=1}^{\infty} \frac{D_i}{(1+r)^i}, \tag{6}$$

where D_i denotes the dividend at year i. In the case of N annuities periods $T_1, \dots, T_N$ with respective growth rate $g_1, \dots, g_N$ followed by a perpetuity with growth rate g_{N+1} the value of a share can be written as

$$P_0 = \sum_{i_1=1}^{T_1} \frac{D_0(1+g_1)^{i_1}}{(1+r)^{i_1}} + \sum_{i_2=T_1+1}^{T_1+T_2} \frac{D_0(1+g_1)^{T_1}(1+g_2)^{i_2-T_1}}{(1+r)^{i_2}} + \cdots$$

$$+ \sum_{i_k=ST_{k-1}+1}^{ST_k} \frac{D_{ST_{k-1}}(1+g_k)^{i_k-ST_{k-1}}}{(1+r)^{i_k}} + \cdots + \sum_{i_N=ST_{N-1}+1}^{ST_N} \frac{D_{ST_{N-1}}(1+g_N)^{i_N-ST_{N-1}}}{(1+r)^{i_N}}$$

$$+ \sum_{i_{N+1}=ST_N+1}^{\infty} \frac{D_{ST_N}(1+g_{N+1})^{i_{N+1}-ST_N}}{(1+r)^{i_{N+1}}}.$$

Then,

$$P_0 = \sum_{k=1}^{N} \sum_{i_k=ST_{k-1}+1}^{ST_k} \frac{D_{ST_{k-1}}(1+g_k)^{i_k-ST_{k-1}}}{(1+r)^{i_k}} + \sum_{i_{N+1}=ST_N+1}^{\infty} \frac{D_{ST_N}(1+g_{N+1})^{i_{N+1}-ST_N}}{(1+r)^{i_{N+1}}}$$

$$= \sum_{k=1}^{N} D_{ST_{k-1}} \sum_{l=1}^{ST_k-ST_{k-1}} \frac{(1+g_k)^l}{(1+r)^{l+ST_{k-1}}} + D_{ST_N} \sum_{j=1}^{\infty} \frac{(1+g_{N+1})^j}{(1+r)^{j+ST_N}}$$

$$= \sum_{k=1}^{N} \frac{D_{ST_{k-1}}}{(1+r)^{ST_{k-1}}} \sum_{l=1}^{T_k} \left[\frac{1+g_k}{1+r}\right]^l + \frac{D_{ST_N}}{(1+r)^{ST_N}} \sum_{j=1}^{\infty} \left[\frac{1+g_{N+1}}{1+r}\right]^j. \tag{7}$$

The last line contains two sums of geometric series that can be calculated as follows

$$\sum_{l=1}^{T_k} \left[\frac{1+g_k}{1+r}\right]^l = \sum_{l=0}^{T_k} \left[\frac{1+g_k}{1+r}\right]^l - 1$$

$$= \frac{1-\left(\frac{1+g_k}{1+r}\right)^{T_k+1}}{1-\frac{1+g_k}{1+r}} - 1 = \frac{1+g_k}{1+r}\left[\frac{1-\left(\frac{1+g_k}{1+r}\right)^{T_k}}{1-\frac{1+g_k}{1+r}}\right] = \frac{1+g_k}{r-g_k}\left[1-\left(\frac{1+g_k}{1+r}\right)^{T_k}\right], \tag{8}$$

and

$$\sum_{j=1}^{\infty}\left[\frac{1+g_{N+1}}{1+r}\right]^j = \sum_{j=0}^{\infty}\left[\frac{1+g_{N+1}}{1+r}\right]^j - 1 = \frac{1}{1-\frac{1+g_{N+1}}{1+r}} - 1 = \frac{1+g_{N+1}}{r-g_{N+1}}. \tag{9}$$

Substituting equations (8) and (9) in (7) we obtain

$$P_0 = \sum_{k=1}^{N} \frac{D_{ST_{k-1}}}{(1+r)^{ST_{k-1}}} \frac{1+g_k}{r-g_k}\left[1-\left(\frac{1+g_k}{1+r}\right)^{T_k}\right] + \frac{D_{ST_N}}{(1+r)^{ST_N}} \frac{1+g_{N+1}}{r-g_{N+1}},$$

$$P_0 = \sum_{k=1}^{N+1} \frac{D_{ST_{k-1}+1}}{(1+r)^{ST_{k-1}}(r-g_k)} A_k,$$

where A_k, for $1 \le k \le N+1$, are given by equations (3) and (4) in section two. This ends the underlying proof.